Sheaves

Sheafification

Input presheaf F on X (top space)

Output sheaf F^{sh}, $F \to F^{sh}$

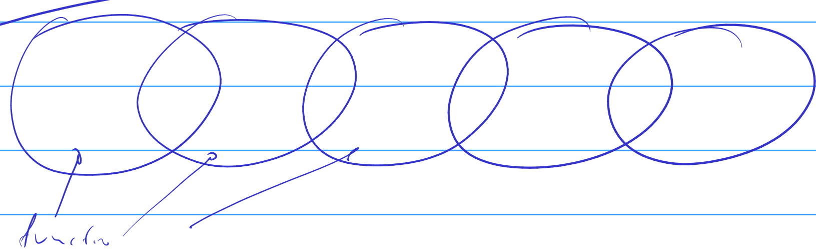
On $U \subset X$ open

$$F^{sh}(U) = \bigcup_{\substack{\text{all coverings of} \\ U \text{ by } \{V_\lambda\}_{\lambda \in \Lambda}}} \{ f_\lambda \in F(V_\lambda) \mid \forall \lambda, \mu \quad \rho_{U_\lambda \cap U_\mu, U_\lambda} f_\lambda = \rho_{U_\lambda \cap U_\mu, U_\mu} f_\mu \}$$

equivalence relation

if we have another covering of U by W_μ s.t. $\forall \mu$ $W_\mu \subset V_\lambda$ some λ

$\Rightarrow$ $f_\lambda \in F(V_\lambda)$ is equivalent to $\{ \rho_{W_\mu, V_\lambda}(f_\lambda) \}_\mu$



Another way: Suppose we have a covering

$U = \bigcup_{\lambda \in \Lambda} V_\lambda$ $\qquad$ $U = \bigcup_{\lambda \in \Lambda'} V'_\lambda$

$f_\lambda \in F(V_\lambda)$ $\qquad$ $f'_\lambda \in F(V'_\lambda)$

agree on intersections as in $\qquad$ agree on intersections

$$(f_\lambda) = (f'_\lambda) \quad \text{if} \quad \forall x \in U \ \exists \text{ open } x \in N \subset U$$

s.t. $N \subset V_\lambda$ some λ $\quad N \subset V'_{\lambda'}$ some λ'

and $\rho_{N, V_\lambda} f_\lambda = \rho_{N, V'_{\lambda'}} f'_{\lambda'}$.

Proposition 1) F^{sh} is a sheaf
2) $\forall$ presheaf F
$\forall$ sheaf G

$F \to F^{sh} \to G$

$$Hom(F^{sh}, G) \longrightarrow Hom(F, G)$$ is a bijection.

Assume we have this.

Let's prove that for $\forall\, \varphi : F \to G$ (F, G sheaves)

$\mathrm{Coker}\, \varphi = (U \to \mathrm{coker}\, \varphi_U : F(U) \to G(U))^{sh}$.

$$G \to \underbrace{(U \to \mathrm{coker}\, \varphi_U)}_{\mathrm{coker}_{pre} \varphi} \to \mathrm{Coker}\, \varphi \to W$$

we need to check : $\forall\, W$

We want:

$Hom(\mathrm{Coker}\, \varphi, W) \to Hom(G, W)$ is injective, the image is those $G \to W$ which become 0 when composed with $\varphi : F \to G$.

to see:

$Hom(\mathrm{Coker}_{pre}\, \varphi, W)$ by Prop is a bijection.

each

$$(\mathrm{Coker}_{pre}\, \varphi)(U) = \frac{G(U)}{\mathrm{Im}\, \varphi_U : F(U) \to G(U)}$$

1) injectivity is clear.

$F \to G \to \mathrm{Coker}_{pre}\, \varphi$

2) composition is clearly zero. $\Rightarrow$ image of. is contained in the RHS.

Conversely if $G \to W$ becomes 0 when composed with φ, then $\exists!$ $\mathrm{coker}_{pre}\, \varphi \to W$

Simply on U $(\mathrm{Coker}_{pre}\, \varphi)(U) \to W(U)$ there $\exists$ unique such map.

$(\mathrm{Coker}_{pre}\, \varphi)(U) = \mathrm{coker}\, \varphi_U$

Since it is unique all diagrams will be commutative to get a map $\mathrm{Coker}_{pre}\,\varphi \to W$.
as required.

Next

Main idea to implement geometric constructions via functors between categories of sheaves.

Suppose $f : X \to Y$ is a continuous map.
Let F be a sheaf on X. Then
$(f_* F)(U) = F(f^{-1}(U))$ is called the direct image sheaf.

Prop $(f_* F)$ is a sheaf.

Proof

- 1) $f_* F$ is a presheaf

$V \subset U$ restrictions:

$$\begin{array}{ccc} (f_* F)(U) & = & F(f^{-1}(U)) \\ \downarrow & & \downarrow \\ (f_* F)(V) & = & F(f^{-1}(V)) \end{array}$$

2) sheaf axiom
$\forall\ U = \bigcup_{\lambda \in \Lambda} V_\lambda$ on Y

$$(*) \quad 0 \to (f_* F)(U) \xrightarrow[1]{} \prod_\lambda (f_* F)(V_\lambda) \xrightarrow[2]{} \prod_{\lambda,\mu} (f_* F)(V_\lambda \cap V_\mu)$$

need to check: 1 is the kernel of 2.
in this case we say that $(*)$ is short exact.

$f^{-1}(U) = \bigcup_{\lambda \in \Lambda} f^{-1}(V_\lambda)$, so $(*)$ is simply the same sequence for F and this covering on X.
so $(*)$ is exact

$\boxtimes$

$$f : X \to Y$$

Inverse image — Suppose G is a sheaf on Y, we construct $f^{-1}(G)$

pick $U \subset X$ open

$$(f^{-1}_{pre} G)(U) = \text{"}\lim_{V \supset f(U)} G(V)\text{"} \qquad \bigcup_{V \supset f(U)} G(V) \Big/ \text{equivalence}$$

$\alpha \in G(V)$ $\alpha = \beta$ if

$\beta \in G(V')$ $R_{V \cap V'; V}\, \alpha = R_{V \cap V'; V'}\, \beta$

Example — $X = \text{point}$, $x \in Y$, G = structure sheaf

$$(f^{-1}_{pre} G)(X) = \text{stalk } \mathcal{O}_{Y,x}.$$

Def $f^{-1} G = (f^{-1}_{pre} G)^{sh}$. $f : X \to Y$

Is it easier to describe $\mathrm{Hom}(f^{-1}G, F)$ for arbitrary sheaf F on X?

$$\mathrm{Hom}((f^{-1}_{pre} G)^{sh}, F) = \mathrm{Hom}(f^{-1}_{pre} G, F) =$$

$\forall U \subset X$ $\lim_{V \supset U} G(f(V)) \to F(U)$

$$\mathrm{Hom}(G, f_* F).$$

$\mathrm{Hom}(f^{-1}G, F) = \mathrm{Hom}(G, f_* F)$ in such situations f^{-1}, f_* are called adjoint functors.

Examples — Banach

$$X \xrightarrow{f} Y$$

assume f injective $\mathrm{Coker}\, f = Y / \overline{\mathrm{Im}\, f}$

$\mathrm{Ker}\, f = \{0\}$ $\mathrm{Coker}\, f = Y \to Y/\overline{\mathrm{Im}\, f}$

$\mathrm{Coker}\, \mathrm{Ker}\, f = X$ $\mathrm{Ker}\, \mathrm{Coker}\, f = \overline{\mathrm{Im}\, f}$

may be $X \neq \overline{X}$ in Y.

fix N

In algebra: filtered vector spaces.

Objects are (V vector space, $F^0 = V \supset F^1 \supset \dots \supset F^N = \{0\}$)

subspaces.

Morphisms $(V, F^\cdot) \to (W, G^\cdot)$ are linear maps preserving filtrations. $f: V \to W$ $f(F^i) \subset G^i$.

given $f: V \to W$

$\operatorname{Ker} f \subset V$ together with the filtration $\operatorname{Ker} f \cap F^i$ is a kernel in this category.

$\operatorname{Coker} f$ $W \to W/\operatorname{Im} f$, filtration: $G^i + \operatorname{Im} f / \operatorname{Im} f$

$\operatorname{Coker} \operatorname{Ker} f = \operatorname{Im} f$, filtration coming from F $\subset$ $\operatorname{Ker} \operatorname{Coker} = \dots$ filtration coming from G.

in general the inclusion is strict.

Examples of sheaves

Which of the following are sheaves on $\mathbb{R}$?

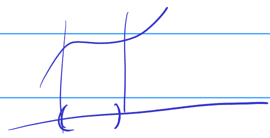
5 minutes ~ 11

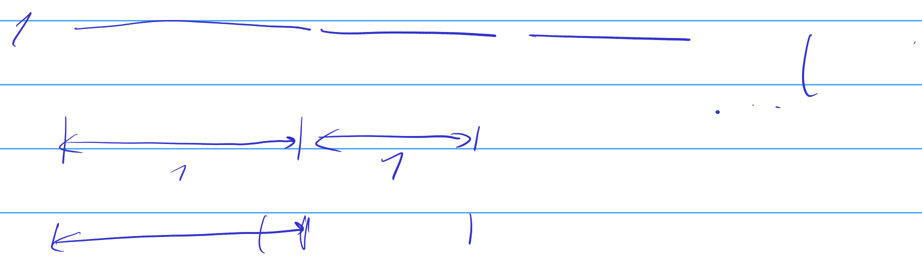
Yes 1) $F(U) =$ C^∞-functions on U

No 2) $F(U) =$ L^2-functions on U (square integrable)

No 3) $F(U) =$ Continuous functions with compact support

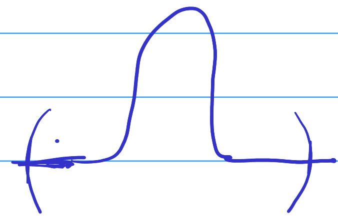
not a presheaf





In 2) sheafification = locally L_2-functions.

In 3) C



Constant sheaf fix abelian group A.

$U \neq \emptyset$

Let $F(U) = A$ $r_{U,V} = Id$

$F(\emptyset) = \{0\}$

(U) (V)

$U \cap V = \emptyset \Rightarrow F(U \cup V) = F(U) \times F(V)$

cannot have A.

$\emptyset = \bigcup_{\lambda \in \emptyset} U_\lambda$

if X is irreducible this is a sheaf.

$\underbrace{F(\emptyset) \subset \prod_{\lambda \in \emptyset} F(U_\lambda)}_{\{0\}}$

$\mathrm{Hom}(A, \prod_{\lambda \in \emptyset} F(U_\lambda)) = \prod \mathrm{Hom}$

Homework

t

$S^1 = [0,1]/_{0 \sim 1}$

$$C^\infty(S^1, \mathbb{R}) \xrightarrow{\frac{d}{dt}} C^\infty(S^1, \mathbb{R})$$

$\parallel$

periodic functions

Compute ker, coker.