Groups today: $U(1)$ $SU(n)$

$$U(1) = \{ x \in \mathbb{C} \mid x\bar{x} = 1 \}$$

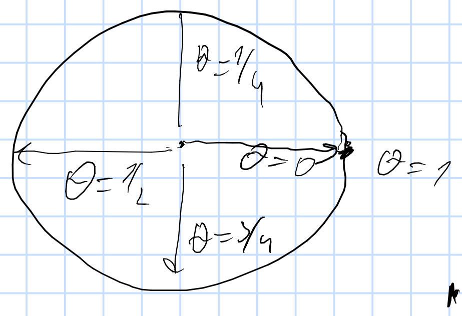
$$\Updownarrow$$

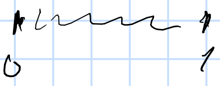
$$|x| = 1$$

$$\Updownarrow$$

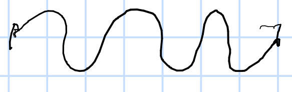
$$x = e^{2\pi i \theta}$$

$\theta \in \mathbb{R}$ (up to $\mathbb{Z}$-shifts)





Fourier analysis:
decompose functions on $[0,1]$ into infinite sums of harmonics of the form $\cos(2\pi\theta k)$ $\sin(2\pi\theta k)$ $k \in \mathbb{Z}$.



$$e^{2\pi i \theta k} = \cos(2\pi\theta k) + \sin(2\pi\theta k) \cdot i$$

equivalently, we can decompose into functions $e^{2\pi i \theta k}$ $k \in \mathbb{Z}$.

Define hermitian form on continuous functions $f \in C(U(1))$ by

$$(f, g) = \int_0^1 f(\theta)\overline{g(\theta)}\, d\theta$$

Let $|f|_2 = \sqrt{(f,f)}$ "L_2-norm"

Let $L_2(U(1))$ = completion of the space $C(U(1))$ with respect to the L_2-norm. This is a Hilbert space, we can search for a orthonormal basis.

Check: $\{ e^{2\pi i \theta k} \mid k \in \mathbb{Z} \}$ forms an orthonormal basis.

$$\int_0^1 e^{2\pi i k_1 \theta}\, \overline{e^{2\pi i k_2 \theta}}\, d\theta =$$

$$= \int_0^1 e^{2\pi i (k_1 - k_2)\theta}\, d\theta = \begin{cases} 1 & (k_1 = k_2) \\ 0 & (k_1 \neq k_2) \end{cases}$$

linear combinations of these functions are dense in $L_2(U(1))$.
($\rightsquigarrow$ Stone–Weierstrass).

$\Rightarrow$

Any function f can be written as

$$f(\theta) = \sum_{k \in \mathbb{Z}} e^{2\pi i \theta k} \underbrace{\left(f, e^{2\pi i \theta k}\right)}_{\in \mathbb{C}} \cdot$$ in L_2-norm

Idea: Note that $\forall k$ $x \in U(1) \mapsto x^k$ is a 1-dimensional representation of $U(1)$, these are all irreducible reps of $U(1)$, so we can guess that for arbitrary compact group something like this happens.

Consider a more general compact group, say $SU(n) = \left\{ \begin{matrix} n\times n \text{ matrices} \\ X \text{ s.t. } X^*X = 1 \end{matrix} \right\}$ $\det X = 1$

$$X^* = \overline{X}^t.$$

$G = SU(n)$

Theorem there is a unique Haar measure on G:

measure: $f \in C(G)$ (continuous function)

measure

$\Rightarrow$ we have an integral

$$\int_G f(x)\,dx$$

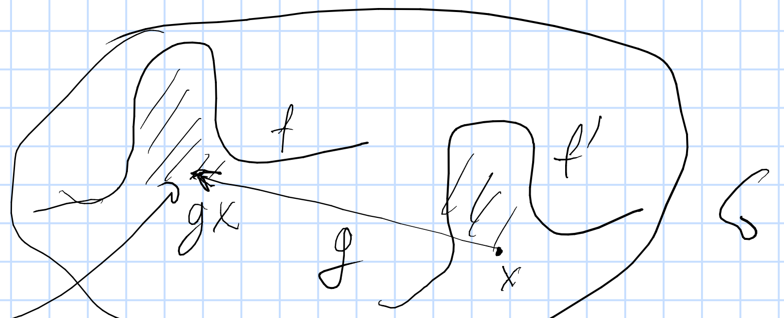
(measure is a linear function from $C(G, \underset{\text{real-valued}}{\mathbb{R}})$ to $\mathbb{R}$)

Satisfying:

$$\min_{x\in G} f \le \int_G f(x)\,dx \le \max_{x\in G} f$$

which is Left-invariant:

$$\int_G f(x)\,dx = \int_G f(gx)\,dx \quad (\forall g \in G).$$

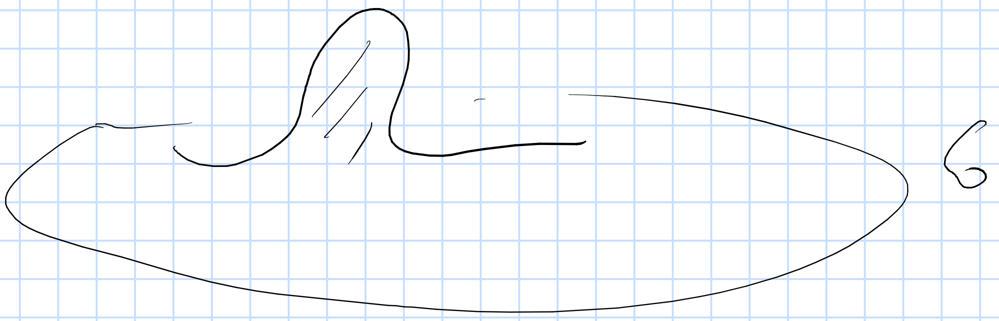


$\int_G f(x)dx$

$$f'(x) = f(gx)$$

Proof "sketch"

Consider f which looks like



$\int f = ?$ try to translate f

by $g_1, g_2, \dots, g_N \in G$ (N large)

so that $\frac{1}{N}\sum_{i=1}^{N} f(g_i x) = \tilde{f}(x)$

Satisfies $\max \tilde{f} - \min \tilde{f} < \varepsilon$.

then $\Rightarrow$

$$\int f(x) = \frac{1}{N} \sum \int f(g_i x)$$

$$= \int \tilde{f} \in [\min \tilde{f}, \max \tilde{f}].$$

$\Rightarrow$ we can compute $\int f(x)$ with precision ε.

Remark The left-invariant measure is also right invariant because $\forall h \in G$

$f \to \int f(xh)$ satisfies all the properties $\left(\int f(gxh) = \int f(xh) \right)$.

$\Rightarrow \int f(xh) = \int f(x)$.

Remark Our measure satisfies

$\int 1 = 1$ (= called normalized measure
= is a probability distribution)

Rep theory

Maschke's theorem:

indecomposable $\Rightarrow$ irreducible

For compact groups:

Lemma

Let's prove that any f. dim rep is Unitarizable

Consider (V, ρ_V) $\rho_V(g): V \to V$ $g \in G$,

ρ_V continuous G compact.

Choose an arbitrary positive definite scalar product $(\cdot,\cdot)'$ on V

Define $(x,y) = \int_G (gx, gy)' \, dg$

Then 1) $(hx, hy) = \int_G (ghx, ghy)' dg$

if $f(g) = (gx, gy)'$

$f'(g) := f(gh) = (ghx, ghy)'$

Since the measure is right-invariant, we obtain $\int f = \int f' \Rightarrow$

$(hx, hy) = (x, y)$.

2) $(\ ,\ )$ is linear, satisfies $(x,y) = \overline{(y,x)}$.

3) $(\ ,\ )$ is positive definite:

Consider $(x,x) = \int_G (gx, gx) \, dg$

the function $f(g) = (gx, gx)'$ is continuous on G, G compact, so

$\min f(g) = f(g_0)$ some g_0

$= (g_0 x, g_0 x) > 0 \Rightarrow$

$(x,x) \geq \min f(g) = (g_0 x, g_0 x) > 0$.

Proof of Maschke:

$V_1 \subset V_2$ subrepresentation $\Rightarrow$

we need?

$\exists$ $V_3 \subset V_2$ another subrepresentation so that $V_2 = V_1 \oplus V_3$.

Pick $V_3 = V_1^\perp$!

1) Schur's Lemma: (V_1,ρ_{V_1}) (V_2,ρ_{V_2}) irreducible

if $f: (V_1,\rho_{V_1}) \to (V_2,\rho_{V_2})$,

$f \neq 0$

satisfies $f(\rho_{V_1}(g)x) = \rho_{V_2}(g) f(x)$

then V_1 isomorphic to V_2

Proof: the same as before

2) $f: (V_1,\rho_{V_1}) \to (V_1,\rho_{V_1})$

Satisfies the same properties $\Rightarrow$

$f = \lambda\, Id$, sor $\lambda \in \mathbb{C}$.

Proof: —— || ——.

Orthogonality

for (V_1,ρ_{V_1}) Let $\chi_{V_1}: G \to \mathbb{C}$

defined by $\chi_{V_1}(g) = Tr\, \rho_{V_1}(g)$.

Then for irreducible $(V_1,\rho_{V_1}), (V_2,\rho_{V_2})$

we have

$$\int_G \chi_{V_1}(g)\overline{\chi_{V_2}(g)}\,dg = \begin{cases} 0 & V_1 \not\cong V_2 \\ 1 & V_1 \cong V_2 \end{cases}$$

Proof Consider $Hom_G(V_1,V_2) \subset Hom(V_1,V_2)$

$\dim Hom_G(V_1,V_2) = 0$ or 1.

Let $\pi: Hom(V_1,V_2) \to Hom(V_1,V_2)$

be defined by

$$\pi(M) = \int_G \underbrace{\rho_{V_2}(g)\, M\, \rho_{V_1}(g)^{-1}}_{\text{matrix for each } g}$$

1) if $M \in Hom_G(V_1,V_2) \Rightarrow$

$\pi(M) = M$

2) for all M $\pi(M) \in Hom_G(V_1,V_2)$

because $\forall h \in G$

$$\rho_{V_2}(h)\pi(M) = \Big(\underbrace{\int \rho_{V_2}(hg)\, M\, \rho_{V_1}(hg)^{-1}}_{\pi(M)}\Big)\rho_{V_1}(h)$$

as before:

$$\dim Hom_G(V_1,V_2) = Tr\,\pi$$

$$Tr\,\pi = \int_G Tr\big(M \to \underbrace{\rho_{V_2}(g)}\, M\, \underbrace{\rho_{V_1}(g)^{-1}}\big)$$

$$= \int_G \underbrace{Tr(\rho_{V_2}(g))}_{\chi_{V_2}(g)}\; Tr(\rho_{V_1}(g)^{-1})$$

Remains to show:

$$Tr(\rho_{V_1}(g)^{-1}) = \overline{Tr\,\rho_{V_1}(g)}.$$

Indeed: $U = \rho_{V_1}(g)$ unitary $\Rightarrow U^*U = Id$

$U^{-1} = U^* \Rightarrow$

$Tr\,U^{-1} = \overline{Tr\,U}$ $\blacksquare$

$$Tr\,\pi = \int \chi_{V_2}(g)\overline{\chi_{V_1}(g)} = \int \chi_{V_1}(g)\overline{\chi_{V_2}(g)}$$

$\det \rho_V(g) = ?$

M matrix $\rightsquigarrow$ char. polynomial $\det(1-\lambda M)$

$\rightsquigarrow tr(M)$

$tr(M^2), \ldots$

There is a relation:

$$\det(1-\lambda M) = e^{-\sum \frac{tr(M^k)}{k}\lambda^k},$$

information of the coeffs of char poly (including det) $\longleftrightarrow$ traces of powers M^k.

$M = \rho_V(g)$

$M^k = \rho_V(g^k)$

Question g a random element of $SU(2)$. $g \sim \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}$.

$|u| = 1$

How is $\operatorname{tr} g$ distributed?

-2 ⊢——⊣ 2 .

Suggestion
Let's look at representations of $SU(2)$.

first: trivial rep.
second: $SU(2)$ acts on $\mathbb{C}^2$ by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(2)$ $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

Clearly irreducible.

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(2) \Rightarrow$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \bar a & \bar c \\ \bar b & \bar d \end{pmatrix} = \mathrm{id} \Rightarrow$$

$\det = 1$

$$g = \begin{pmatrix} a & b \\ -\bar b & \bar a \end{pmatrix} \qquad |a|^2 + |b|^2 = 1$$

$a, b \in \mathbb{C}$.

$a = a_1 + i a_2$
$b = b_1 + i b_2$
$a_1^2 + b_1^2 + a_2^2 + b_2^2 = 1$

$SU(2) \cong S^3$.

More representations:
fix $k \geq 1$
Let Sym^k be the space of homogeneous degree k polynomials in x, y. $a_0 x^k + a_1 x^{k-1} y + \dots$

$SU(2)$ acts via
$$f(x,y) \Rightarrow f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}\begin{pmatrix} x \\ y \end{pmatrix}\right).$$

we obtain a sequence of reps.
These are all representations.

Characters:
V is a rep,
$$\chi_V\left(\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}\right) = ?$$

$|u| = 1$

$$\mathrm{Sym}^k \rightsquigarrow \underbrace{u^k + u^{k-2} + \dots + u^{-k}}_{\displaystyle = \chi_{\mathrm{Sym}^k}\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}}.$$

This theory implies that χ_{Sym^k} are orthonormal.